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NOTE ON ANHARMONIC RATIOS.

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THE present note is a continuation of my paper "New Notation for Anharmonic Ratios", p. 185, ANALYST, Vol. IX; and the sections and equations are for convenience numbered consecutively with those of that paper.

12. As in section 2, the anharmonic ratio which we have denoted by

$$x = \begin{matrix} P & A \\ B & Q \end{matrix}$$

is regarded as the coordinate of P in an anharmonic system in which the coordinates of A, B and Q, which we may call *the fundamental points*, are 0, ∞, and 1, respectively. Now if p, a, b and q denote ordinary coordinates of the same points, that is, their distance from an assumed origin measured with an assumed unit of length, we may also write

$$x = \begin{matrix} p & a \\ b & q \end{matrix},$$

in which p, a, b and q are numbers; and, in accordance with the definition of the symbol in section 3, we have

$$x = \begin{matrix} p & a \\ b & q \end{matrix} = \frac{(p-a)(b-q)}{(p-b)(a-q)}. \qquad (7)$$

13. The results of sections 4, 7, and 8 may then be expressed as follows:

1°. If, in this symbol, we interchange a and b, we convert the coordinate x into its reciprocal, interchanging the fundamental points 0 and ∞, and leaving the point 1 unmoved.

2°. If we interchange a and q, we convert x into its complement, interchanging the fundamental points 0 and 1, and leaving the point ∞ unmoved.

3°. If we interchange b and q, we take the conjugate, interchanging the points 1 and ∞, and leaving the point 0 unmoved. Thus, if

$$x = \begin{matrix} p & a \\ b & q \end{matrix}, \qquad Rx = \frac{1}{x} = \begin{matrix} p & b \\ a & q \end{matrix},$$

$$Cx = 1-x = \begin{matrix} p & q \\ b & a \end{matrix}, \qquad Jx = \frac{x}{x-1} = \begin{matrix} p & a \\ q & b \end{matrix}$$

In like manner we have

$$CRx = \frac{x-1}{x} = \begin{matrix} p & q \\ a & b \end{matrix}, \qquad RCx = \frac{1}{1-x} = \begin{matrix} p & b \\ q & a \end{matrix},$$

in which the fundamental points are cyclically interchanged.

14. We have three special varieties of coordinates.

I. When the fundamental point B is at an infinite distance; the coordi nates become ordinary coordinates or distances measured from A, AQ being the unit of length.

II. When the fundamental point A is at infinity; the coordinates become the reciprocals of the distances from B, BQ being the unit of length.

III. When the fundamental point Q is at infinity; the coordinates become the "position ratios" of section 1.

In transforming the system I to the system II, the operation is R; the points A and B interchanging: in transforming I to III, the operation is J; B and Q interchanging: in transforming II to III, the operation is C; A and Q interchanging.

15. The symbol $\begin{matrix} p & a \\ b & q \end{matrix}$ is obviously, from equation (7), a function of p, a, b and q, whose value is unaltered by adding the same quantity to each of the letters, or by multiplying them by the same quantity; that is to say, by changing the position of the origin in assigning the ordinary coordinates of P, A, B and Q, or by changing the scale with which those coordinates are measured, which is equivalent to changing the position of the unit point.

16. But furthermore the value of the symbol is unchanged when if for its elements we substitute any anharmonic coordinates of the same points; or

$$\begin{matrix} x & y \\ z & u \end{matrix} = \begin{matrix} x' & y' \\ z' & u' \end{matrix}, \tag{8}$$

if

$$x = \begin{matrix} x' & a \\ b & q \end{matrix} \qquad y = \begin{matrix} y' & a \\ b & q \end{matrix} \qquad \text{etc.}$$

For, since $x = \dfrac{(x'—a)\,(b—q)}{(x'—b)\,(a—q)}$ etc., we may write

$$x = \frac{\alpha x' + \beta}{\gamma x' + \delta}, \qquad y = \frac{\alpha y' + \beta}{\gamma y' + \delta}, \quad \text{etc.,} \tag{9}$$

whence

$$\begin{aligned} x — y &= \frac{\alpha\delta x' + \beta\gamma y' — \alpha\delta y' — \beta\gamma x'}{(\gamma x' + \delta)\,(\gamma y' + \delta)} \\ &= \frac{(\alpha\delta — \beta\gamma)\,(x' — y')}{(\gamma x' + \delta)\,(\gamma y' + \delta)}. \end{aligned}$$

Employing like expressions for the other differences, we have

$$\begin{matrix} x & y \\ z & u \end{matrix} = \frac{(x—y)\,(z—u)}{(x—z)\,(y—u)} = \frac{(x'—y')\,(z'—u')}{(x'—z')\,(y'—u')} = \begin{matrix} x' & y' \\ z' & u' \end{matrix}.$$

In particular, we may in the equation

$$x = \begin{matrix} p & a \\ b & q \end{matrix}$$

substitute for p, a, b, q the anharmonic coordinates of the points P, A, B and Q in the very system which this equation defines; we thus have

$$x = \frac{x \quad 0}{\infty \quad 1},$$

as may be easily verified by equation (7).

17. It is obvious that in equation (8) both x, y, z, u and x', y', z', u' may be anharmonic coordinates of the same four points referred to different fundamental points; for each member will be equal to the corresponding function of the ordinary coordinates of the same points. In other words, the anharmonic ratio of the anharmonic coordinates of four points, referred to any fundamental points, is constant, for it is simply the anharmonic ratio of the four points. It follows that the relation between the anharmonic coordinates of the same point in different systems is of the form expressed by equation (9).

18. If two lines L and L' be homographically divided, and corresponding points A, B, Q and A', B', Q' be taken as fundamental points; then, by the definition of homographic division, the anharmonic coordinates x and x' of corresponding points P and P' are equal. Hence, if non-corresponding points are taken as fundamental points, x and x' are the same as the anharmonic coordinates of the same point referred to different fundamental points; for x' is the same as the coordinate of P when referred to the points on L which correspond to the fundamental points taken on L'. x and x' have therefore, in this case also, a relation of the form expressed by equation (9).

19. When the lines L and L' coincide, and P and P' are referred to the same fundamental points, the case is equivalent to that last mentioned, since three pairs of corresponding points cannot coincide, unless all corresponding points coincide. It is possible however to make two pairs of corresponding points coincide, and one of these may be a pair selected at random: for if this pair be brought into coincidence (the lines L and L' intersecting at any angle), the corresponding points are in perspective, and a line drawn through the centre of perspective perpendicular to either of the bisectors of the angles between L and L' will determine a pair of corresponding points equidistant from the coincident pair at the intersection. When the lines are revolved into coincidence, one of these pairs of corresponding points comes into coincidence, and we have two pairs of coincident corresponding points, which are in fact the double points of the homographic system.

20. If two of the fundamental points are taken at the double points, the relation between x and x' is simplified. Thus, if A and B are at the double points, so that

$$x = \frac{P \quad A}{B \quad Q}, \qquad x' = \frac{P' \quad A}{B \quad Q} = \frac{P \quad A}{B \quad Q_1},$$

where Q_1, see section 18, is the point of L corresponding to Q regarded as a point of L'; then the definition of x, equation (7), shows that the relation is of the form

$$x' = mx. \tag{10}$$

This is the simplest relation possible, being the form assumed by (9) when $\beta = 0$ and $\gamma = 0$; and is such that x and x' vanish simultaneously, and become infinite simultaneously, as should be the case, since P and P' arrive simultaneously at A, and also simultaneously at B.

Equation (10) is by equation (1) equivalent to

$$\frac{AP'}{BP'} = m\,\frac{AP}{BP},$$

whence

$$\begin{matrix} P & A \\ B & P' \end{matrix} = \frac{1}{m},$$

that is, the anharmonic ratio of two corresponding points and the double points, is constant.

21. If however we take B and Q at the double points, so that

$$x' = \begin{matrix} P' & A \\ B & Q \end{matrix} = \begin{matrix} P & A_1 \\ B & Q \end{matrix},$$

then the complements of the coordinates,

$$1 - x = \begin{matrix} P & Q \\ B & A \end{matrix}, \quad \text{and} \quad 1 - x' = \begin{matrix} P & Q \\ B & A_1 \end{matrix},$$

have the relation given in equation (10); that is,

$$1 - x' = m(1 - x),$$

or

$$x' = 1 - m + mx, \tag{11}$$

a relation of the form (9), in which $\gamma = 0$ and $\alpha + \beta = \delta$; and in which x and x' take the value unity simultaneously, and become infinite simultaneously.

22. Again, if A and Q are taken at the double points, so that

$$x' = \begin{matrix} P' & A \\ B & Q \end{matrix} = \begin{matrix} P & A \\ B_1 & Q \end{matrix},$$

then the conjugates of the coordinates,

$$\frac{x}{x-1} = \begin{matrix} P & A \\ Q & B \end{matrix}, \quad \text{and} \quad \frac{x'}{x'-1} = \begin{matrix} P & A \\ Q & B_1 \end{matrix},$$

have the relation given in (10); thus

$$\frac{x'}{x'-1} = m\,\frac{x}{x-1},$$

whence

$$x' = \frac{mx}{(m-1)x+1}, \tag{12}$$

a relation of the form (9), in which $\beta=0$ and $\alpha=\gamma+\delta$; and in which x and x' vanish simultaneously, and take the value unity simultaneously.

23. When two points of L, whose distance is the same as that of their corresponding points on L' have been found, as in section 19, the lines L and L' may be made to coincide in such a manner that each of these points falls upon the point corresponding to the other. If now the fundamental points A and B are taken at these *doubly corresponding* points; in other words, if A as a point of L corresponds with B as a point of L', and A as a point of L' corresponds with B as a point of L, we have

$$x = \frac{P\ \ A}{B\ \ Q}, \qquad x' = \frac{P'\ \ A}{B\ \ Q} = \frac{P\ \ B}{A\ \ Q_1};$$

since, as in section 18, x' is the same as the coordinate of P, referred to the points of L corresponding to the fundamental points A, B, Q regarded as points of L'. Hence x' is now the reciprocal of the x' in equation (10), and the relation is of the form

$$xx' = k, \tag{13}$$

in which $x=\infty$ gives $x'=0$.

24. In like manner, if B and Q are at the doubly corresponding points,

$$x' = \frac{P'\ \ A}{B\ \ Q} = \frac{P\ \ A_1}{Q\ \ B};$$

and x' is the conjugate of its value in equation (11). Replacing it therein by $x' \div (x'-1)$, the relation is of the form

$$xx' = x + x' + k, \tag{14}$$

in which $x=\infty$ gives $x'=1$.

25. Again, if A and Q are at the doubly corresponding points,

$$x' = \frac{P'\ \ A}{B\ \ Q} = \frac{P\ \ Q}{B_1\ \ A},$$

which is the complement of x' in equation (12); and replacing x' by $1-x'$ in that equation, we have a relation of the form

$$1 = x + x' + kxx', \tag{15}$$

in which $x=0$ gives $x'=1$.

26. The infinitely distant points on L and L' may be regarded as always coinciding when L' is superposed upon L; hence, if the points each of which corresponds on one line to the infinitely distant point upon the other line are made to coincide, this point of coincidence and the point at infinity are doubly corresponding points. Hence B, in equation (13), may be at infinity; but in this case the coordinates become distances from A. The two ranges of points now form a system in involution; equation (13) expressing that the product of the distances of corresponding points from A, the centre of involution, is constant.